Trig. Eq^n's

Sol^n

- Particular Sol^n (sol^n with restriction)
- General Sol^n (Sol^n w/o restriction)

While solving a trig. eq^n,

Root Loss or Extraneous root may occur.

A reason for above is Change in Domain of Orig. Eq^n

General Sol^{n}s

1) $\sin(\theta) = \sin(\alpha) \iff \boxed{\theta = n\pi + (-1)^n \alpha}$

Generally, $\alpha \in \left[-\pi/2, \pi/2\right]$ $\quad n \in \mathbb{Z}$.

2) $\cos(\theta) = \cos(\alpha) \iff \boxed{\theta = 2\pi n \pm \alpha}$

Generally, $\alpha \in [0, \pi]$ $n \in \mathbb{Z}$

3) $\tan(\theta) = \tan(\alpha) \iff \boxed{\theta = n\pi + \alpha}$

Generally, $\alpha \in (-\pi/2, \pi/2)$ $n \in \mathbb{Z}$

4) $\left.\begin{array}{l} \sin^2(\theta) = \sin^2(\alpha) \\ \cos^2(\theta) = \cos^2(\alpha) \\ \tan^2(\theta) = \tan^2(\alpha) \end{array}\right\} \iff \boxed{\theta = n\pi \pm \alpha}$

5) $\sin(\theta) = 1 \iff \boxed{\theta = (4n+1)\pi/2}$

$\sin(\theta) = (-1) \iff \boxed{\theta = (4n-1)\pi/2}$

$\sin(\theta) = 0 \iff \boxed{\theta = n\pi}$

6) $\cos(\theta) = 1 \iff \boxed{\theta = 2\pi n}$

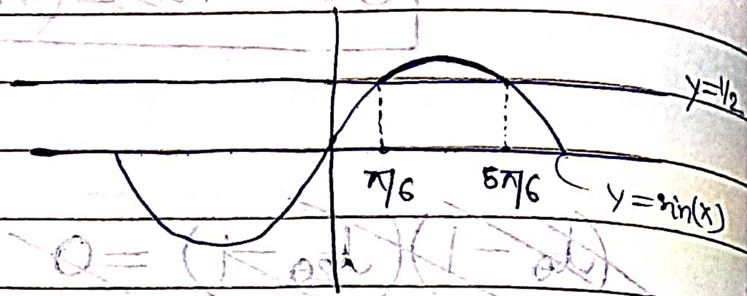
$\cos(\theta) = (-1) \iff \boxed{\theta = (2n+1)\pi}$

$\cos(\theta) = 0 \iff \boxed{\theta = (4n \pm 1)\pi/2}$

Trig. Inequalities

Q) $\sin(x) > 1/2$

A) Draw graph!



$$\Rightarrow \boxed{x \in \left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)}$$

More techenically,

$$\boxed{x \in \bigcup_{n \in \mathbb{Z}} \left(2n\pi + \pi/6, 2n\pi + 5\pi/6\right)}$$

Boundary Cond^{n}s

Q) $\sin(\theta) \geq 1$

A) $\sin(\theta) = 1 \Rightarrow \boxed{\theta = (4n+1)\frac{\pi}{2}}$

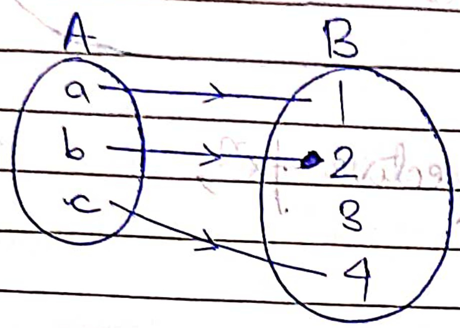
Q) $\sin(x) + \cos(x) = 3/2$

A) $s(x+\pi/4) = 3/2\sqrt{2} >$

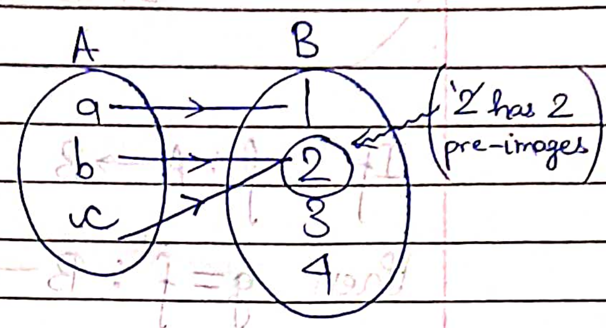
$\Rightarrow \boxed{x \notin \mathbb{R}}$

I.T.F.

One – One



Many-One

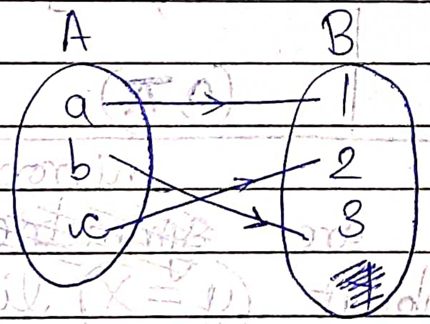


☆ Increasing & Decreasing fx^ns are ALWAYS One-one.

☆ Periodic fx^ns are ALWAYS Many-One.

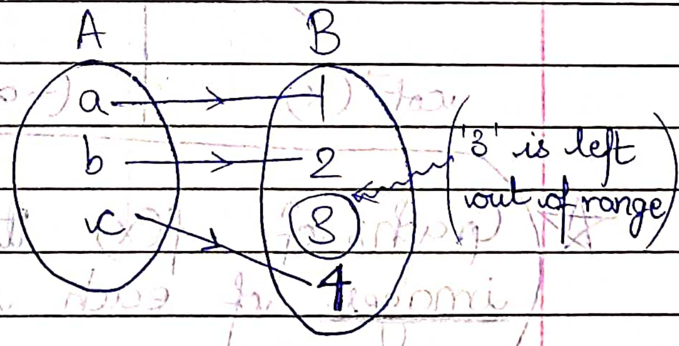
Onto

Range = Codomain



Into

Range ⊂ Codomain



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Inverse Trig. fx^n

☆ For inverse of any fx^n to exist, the fx^n should be

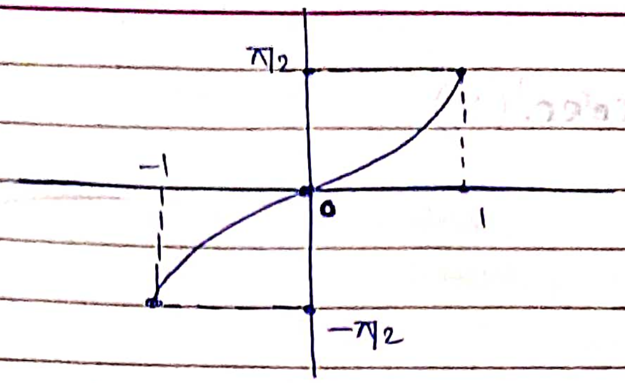
ONE-ONE & ONTO

If $f: A \rightarrow B$ be a bijective fx^n, then $g = f^{-1} : B \rightarrow A$.

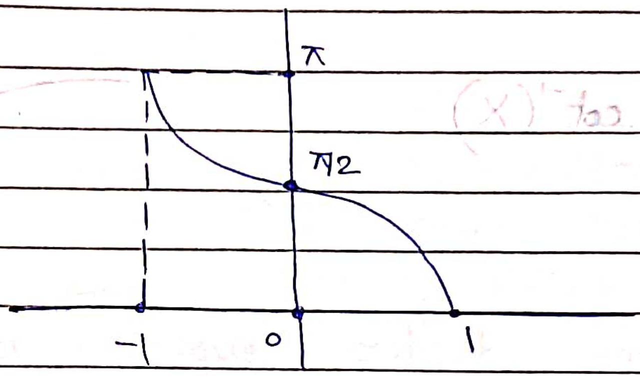
fx^n	Domain	Range
$\sin^{-1}(x)$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\sec^{-1}(x)$	$R-(-1,1)$	$[0,\pi]-\{\pi/2\}$
$\operatorname{cosec}^{-1}(x)$	$R-(-1,1)$	$[-\pi/2, \pi/2]-\{0\}$
$\cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$

☆ Graphs of $f(x)$ & $f^{-1}(x)$ are ~~symmetric~~ mirror images of each other about $y = x$ line.

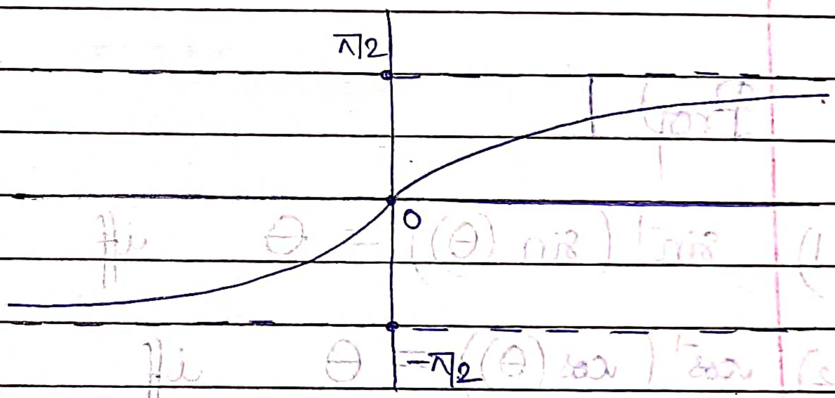
1) $\sin^{-1}(x)$



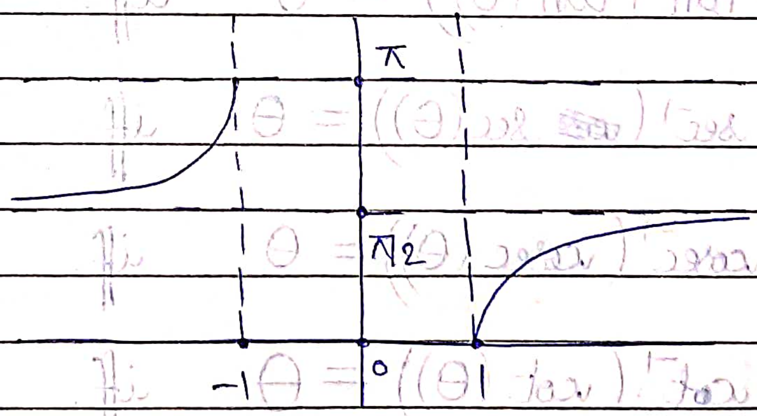
2) $\cos^{-1}(x)$



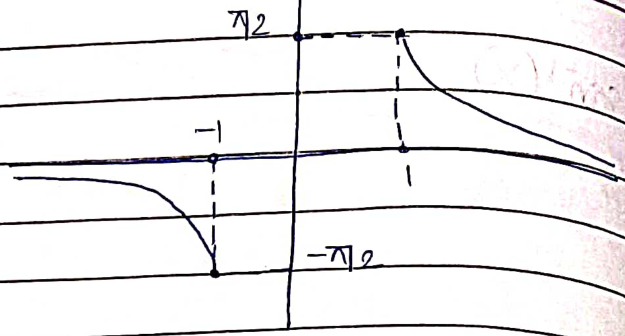
3) $\tan^{-1}(x)$



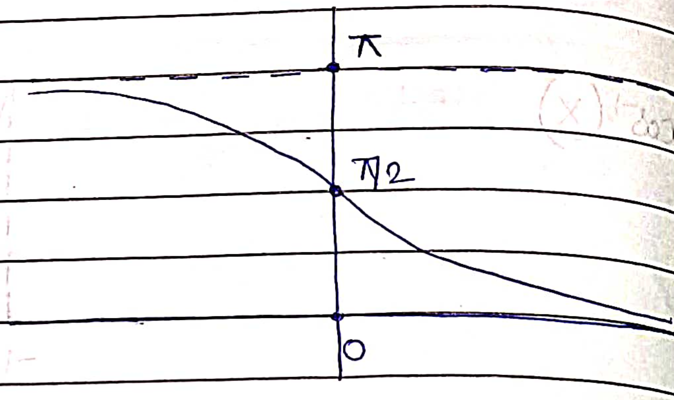
4) $\sec^{-1}(x)$



5) $cosec^{-1}(x)$



6) $cot^{-1}(x)$



Propt 1

1) $\sin^{-1}(\sin(\theta)) = \theta$ iff $\theta \in [-\pi/2, \pi/2]$

2) $\cos^{-1}(\cos(\theta)) = \theta$ iff $\theta \in [0, \pi]$

3) $\tan^{-1}(\tan(\theta)) = \theta$ iff $\theta \in (-\pi/2, \pi/2)$

4) $\sec^{-1}(\sec(\theta)) = \theta$ iff $\theta \in [0, \pi] - \{\pi/2\}$

5) $cosec^{-1}(cosec(\theta)) = \theta$ iff $\theta \in [-\pi/2, \pi/2] - \{0\}$

6) $\cot^{-1}(\cot(\theta)) = \theta$ iff $\theta \in (0, \pi)$

Eg – $\sin^{-1}(\sin(3\pi/4)) \neq 3\pi/4$

$$\sin^{-1}(\sin(3\pi/4)) = \sin^{-1}(\sin(\pi/4)) = \pi/4$$

☆ While solving ANY Q, convert θ so as it lies in the ~~pre~~ principal domain.

Eg – $\sin^{-1}(\sin(12)) = \sin^{-1}(\sin(\underbrace{-4\pi+12}_{\approx(-0.56)\in[-\pi/2,\pi/2]})) = 12-4\pi$

Since, $\sin^{-1}()$ gives unique output, only 1 values that is unique will be obtained.

$$\sin^{-1}(\sin(12)) = \sin^{-1}(\sin(\underbrace{12-3\pi}_{\approx 2.58 \notin [-\pi/2,\pi/2]})) \neq 12-3\pi$$

Eg – $\sin^{-1}(\sin(12)) + \cos^{-1}(\cos(12))$

$$= \sin^{-1}(\sin(\underbrace{12-4\pi}_{\approx(-0.56)\ ✓})) + \cos^{-1}(\cos(\underbrace{12-4\pi}_{\approx(-0.56)\ ✗}))$$

$$= \sin^{-1}(\sin(\underbrace{12-4\pi}_{\approx(-0.56)\ ✓})) + \cos^{-1}(\cos(\underbrace{4\pi-12}_{\approx 0.56\ ✓}))$$

$$= (12-4\pi) + (4\pi-12) = \boxed{0}$$

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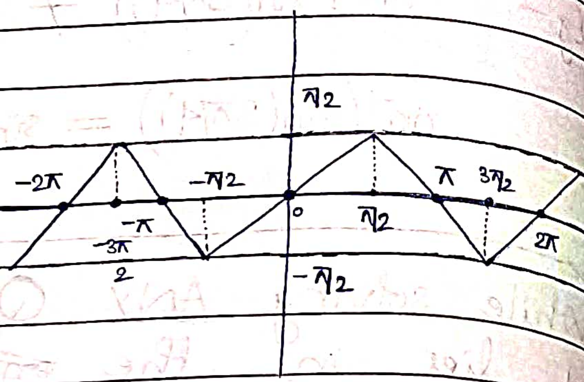
$T^{-1}(T)$ fx^n

1) $f(x) = \sin^{-1}(\sin(x))$

Domain $= \mathbb{R}$
Range $= [-\pi/2, \pi/2]$

Period $= 2\pi$

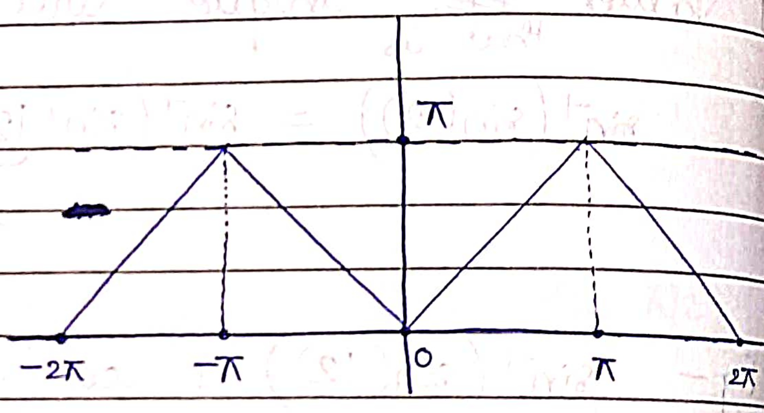


★ Slope of all lines is 1 or −1.
Use intercept & slope to write eq^n quickly.

2) $f(x) = \cos^{-1}(\cos(x))$

Domain $= \mathbb{R}$
Range $= [0, \pi]$

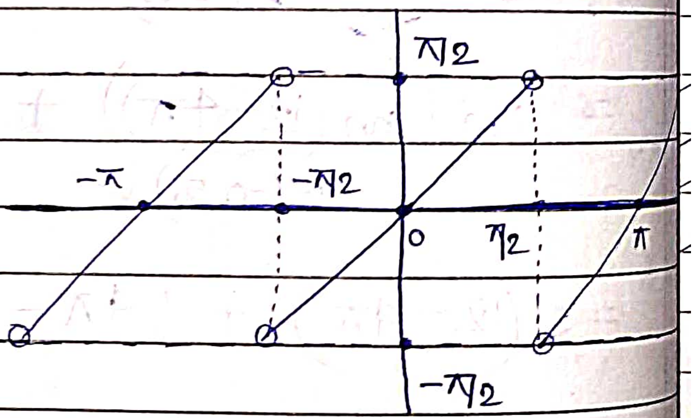
Period $= 2\pi$



3) $f(x) = \tan^{-1}(\tan(x))$

Domain $= \mathbb{R} - \{(2n+1)\pi/2\}$
Range $= (-\pi/2, \pi/2)$

Period $= \pi$



Slopes of all lines is 1.

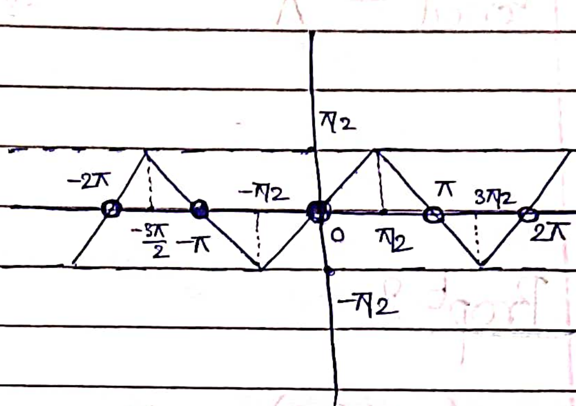
Use intercept & slope to write eq^n quickly.

4) ~~cosec~~ $f(x) = \operatorname{cosec}^{-1}(\operatorname{cosec}(x))$

Domain $= \mathbb{R} - \{n\pi\}$

Range $= [-\pi/2, \pi/2] - \{0\}$

Period $= 2\pi$

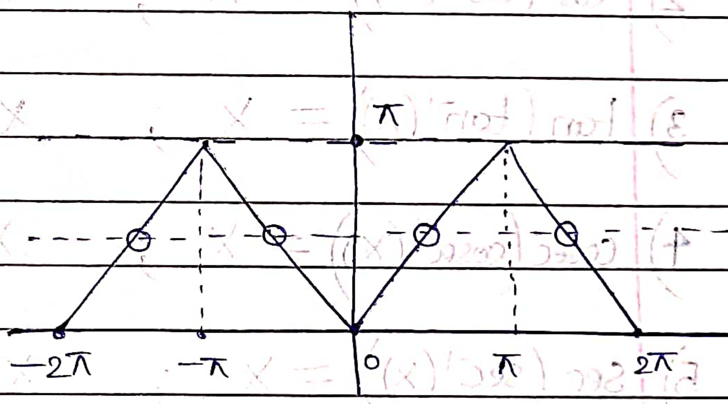


5) $f(x) = \sec^{-1}(\sec(x))$

Domain $= \mathbb{R} - \{(2n+1)\pi/2\}$

Range $= [0, \pi] - \{\pi/2\}$

Period $= 2\pi$



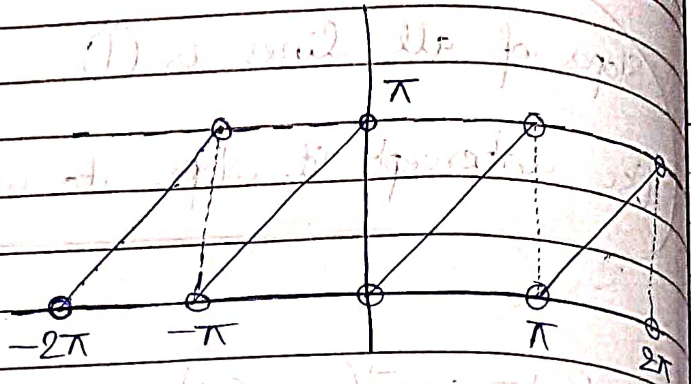
~~6) $f(x) = \cot^{-1}(\cot(x))$~~

~~Domain $= \mathbb{R} - \{n\pi\}$~~

~~Range $= [-\pi/2, \pi/2] - \{0\}$~~

~~Period $= \pi$~~

6) $f(x) = \cot^{-1}(\cot(x))$



Domain $= \mathbb{R} - \{n\pi\}$

Range $= (0, \pi)$ $\cancel{\{\pi/2\}}$

Period $= \pi$

Prop^t 2

1) $\sin(\sin^{-1}(x)) = x \quad ; \quad x \in [-1, 1]$

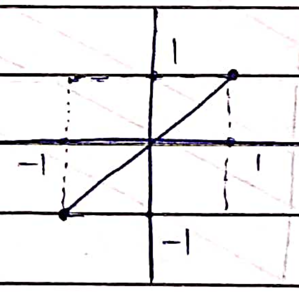
2) $\cos(\cos^{-1}(x)) = x \quad ; \quad x \in [-1, 1]$

3) $\tan(\tan^{-1}(x)) = x \quad ; \quad x \in \mathbb{R}$

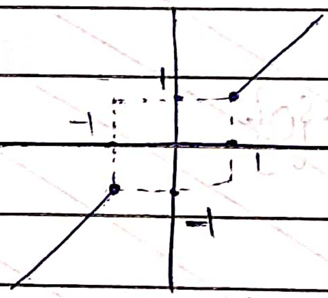
4) $\operatorname{cosec}(\operatorname{cosec}^{-1}(x)) = x \quad ; \quad x \in \mathbb{R} - (-1, 1)$

5) $\sec(\sec^{-1}(x)) = x \quad ; \quad x \in \mathbb{R} - (-1, 1)$

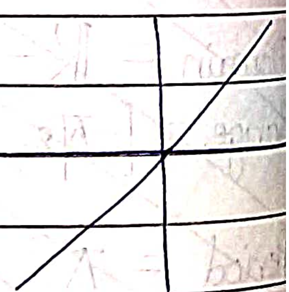
6) $\cot(\cot^{-1}(x)) = x \quad ; \quad x \in \mathbb{R}$



1), 2)



4), 5)



3), 6)

Propt 3

1) $\sin^{-1}(x) + \cos^{-1}(x) = \pi/2$; $|x| \leq 1$

2) $\tan^{-1}(x) + \cot^{-1}(x) = \pi/2$; $x \in \mathbb{R}$

3) $\sec^{-1}(x) + \operatorname{cosec}^{-1}(x) = \pi/2$; $|x| \geq 1$

Proof: 1) Let $\theta = \sin^{-1}(x)$ where $|x| \leq 1$
& $|\theta| \leq \pi/2$

$\Rightarrow \sin(\theta) = x$

$\Rightarrow \cos\left(\pi/2 - \theta\right) = x$

If $(\pi/2 - \theta) \in [0, \pi]$ i.e. $\theta \in [-\pi/2, \pi/2]$, then

$(\pi/2 - \theta) = \cos^{-1}(x)$

Hence, $\theta + (\pi/2 - \theta) = \sin^{-1}(x) + \cos^{-1}(x)$

$\Rightarrow \sin^{-1}(x) + \cos^{-1}(x) = \pi/2$

Similarly for others

Q) Find x if $\sin^{-1}(x) + \tan^{-1}(x) + \operatorname{cosec}^{-1}(x) + \sec^{-1}(x) + \cot^{-1}(x) + \cos^{-1}(x) = 3\pi/2$.

A) $(|x| \leq 1) \cap (x \in \mathbb{R}) \cap (|x| \geq 1) \Rightarrow x = \pm 1$

Prop^t 4

1) $\sin^{-1}(-x) = -\sin^{-1}(x)$

2) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

3) $\tan^{-1}(-x) = -\tan^{-1}(x)$

4) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$

5) $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$

6) $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

Proof : 1) Let $\theta = \sin^{-1}(-x)$, where $|x| \leq 1$ & $|\theta| \leq \pi/2$

$$\Downarrow$$

$$\sin(\theta) = (-x)$$

$$\Rightarrow \sin(-\theta) = x$$

$$\cancel{\Rightarrow -\theta = \sin^{-1}(x)}$$

If $(-\theta) \in [-\pi/2, \pi/2]$ i.e. $\theta \in [-\pi/2, \pi/2]$, then

$$(-\theta) = \sin^{-1}(x)$$

$$\Rightarrow \sin^{-1}(-x) = -\sin^{-1}(x)$$

Similarly for ~~others~~ 3), 4).

2) Let $\theta = \cos^{-1}(-x)$; where $|x| \leq 1$ & $\theta \in [0, \pi]$

$\Rightarrow \cos(\theta) = -x$

$\Rightarrow \cos(\pi - \theta) = x$

If $(\pi - \theta) \in [0, \pi]$ i.e. $\theta \in [0, \pi]$, then

$$(\pi - \theta) = \cos^{-1}(x)$$

$$\Rightarrow \cos^{-1}(-x) = \pi - \cos^{-1}(x)$$

Similarly for ~~others~~ 5), 6).

Prop^t 5

1) $\sin^{-1}(x) = \tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\dfrac{1}{x}\right)$

~~2) $\cos^{-1}(x) = \tan^{-1}$~~

2) $\sin^{-1}(x) = \cos^{-1}\left(\sqrt{1-x^2}\right) = \cot^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) = \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$; $x > 0$

3) $\cos^{-1}(x) = \sin^{-1}\left(\sqrt{1-x^2}\right) = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) = \operatorname{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$; $x > 0$

4) $\tan^{-1}(x) = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \operatorname{cosec}^{-1}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$

6) $\cot^{-1}(x) = \cos^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \sec^{-1}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$

☆ 5) $\tan^{-1}(x) = \begin{cases} \cot^{-1}(1/x) ; & x > 0 \\ \cot^{-1}(1/x) - \pi ; & x < 0 \end{cases}$

☆ 7) $\cot^{-1}(x) = \begin{cases} \tan^{-1}(1/x) ; & x > 0 \\ \pi + \tan^{-1}(1/x) ; & x < 0 \end{cases}$

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Proof: 5) Let $\theta = \tan^{-1}(x)$; where $|\theta| \leq \pi/2$

$\Rightarrow \tan(\theta) = x \Rightarrow \cot(\theta) = 1/x$

C1: If $\theta > 0 \Rightarrow$ ~~cot~~ $\theta \in (0, \pi)$

$\Rightarrow \theta = \cot^{-1}(1/x) \Rightarrow \tan^{-1}(x) = \cot^{-1}(1/x)$

C2: If $\theta < 0 \Rightarrow (\theta + \pi) \in (0, \pi)$

$\Rightarrow (\theta + \pi) = \cot^{-1}(1/x) \Rightarrow \tan^{-1}(x) = (\cot^{-1}(1/x) - \pi)$

Similarly for 7).

Formulae

1) $$\sin^{-1}(x) + \sin^{-1}(y) = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right); & x \geq 0,\ y \geq 0,\ x^2+y^2 \leq 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right); & x \geq 0,\ y \geq 0,\ x^2+y^2 > 1 \end{cases}$$

Proof: Let $\alpha = \sin^{-1}(x) \Rightarrow x = \sin(\alpha)$

$\beta = \sin^{-1}(y) \Rightarrow y = \sin(\beta)$

where $x, y \geq 0$ where $\alpha, \beta \in [0, \pi/2]$

Now, $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
$= \left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

C1 : $0 \leqslant (\alpha+\beta) \leqslant \pi/2 \Rightarrow (\alpha+\beta) = \sin^{-1}(\text{ww})$

$\Rightarrow \sin^{-1}(x) + \sin^{-1}(y) = \sin^{-1}(\text{ww})$

$\beta \leqslant (\pi/2 - \alpha)$

$\xrightarrow{\sin()} \Downarrow$

$y \leqslant \sqrt{1-x^2}$ $\Rightarrow$ if $x^2+y^2 \leqslant 1$

C2 : $\pi/2 < (\alpha+\beta) \leqslant \pi \Rightarrow [\pi - (\alpha+\beta)] \in$ ~~$[-\pi/2, 0]$~~ $[0, \pi/2]$

$\beta > (\pi/2 - \alpha) \Rightarrow \pi - (\alpha+\beta) = \sin^{-1}(\text{ww})$

$\xrightarrow{\sin()} \Downarrow$

$\Rightarrow \sin^{-1}(x) + \sin^{-1}(y) = \left(\pi - \sin^{-1}(\text{ww})\right)$

$y > \sqrt{1-x^2}$ $\Rightarrow$ if $x^2+y^2 > 1$

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3) $\tan^{-1}(x)+\tan^{-1}(y) = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right) & ; x>0, y>0, xy<1 \\ \pi/2 & ; x>0, y>0, xy=1 \\ \pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right) & ; x>0, y>0, xy>1 \end{cases}$

Proof: Let $\alpha = \tan^{-1}(x) \Rightarrow \tan(\alpha) = x$

$\beta = \tan^{-1}(y) \Rightarrow \tan(\beta) = y$

where $x, y > 0$ where $\alpha, \beta \in [0, \pi/2)$

Now, $\tan(\alpha+\beta) = \left(\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\right)$

$= \left(\frac{x+y}{1-xy}\right)$

C1: $(\alpha+\beta) \in \left(0, \frac{\pi}{2}\right) \Rightarrow (\alpha+\beta) = \tan^{-1}(\text{un})$

$\Downarrow$

$\beta < (\pi/2 - \alpha) \Rightarrow \tan^{-1}(x)+\tan^{-1}(y) = \tan^{-1}(\text{un})$

$\tan() \rightarrow \Downarrow$

$t_\beta < T_\alpha$ if $xy<1$

$\Rightarrow t_\beta t_\alpha < 1$

C2: $(\alpha+\beta) = \pi/2 \Rightarrow \tan^{-1}(x)+\tan^{-1}(y) = \pi/2$

$\Downarrow$

$\Rightarrow \beta = (\pi/2 - \alpha)$ if $xy=1$

$\Rightarrow t_\beta = T_\alpha$

C3: $(\alpha+\beta) \in (\pi/2, \pi) \Rightarrow \left(-\pi + (\alpha+\beta)\right) \in (0, \pi/2)$

$\Downarrow$

$\beta > (\pi/2 - \alpha) \Rightarrow \left(-\pi + (\alpha+\beta)\right) = \tan^{-1}(m)$

$\tan() \longrightarrow \Downarrow$

$\Rightarrow \tan^{-1}(x) + \tan^{-1}(y) = \pi + \tan^{-1}(m)$

$T_\beta > T_\alpha$

if $xy > 1$

~~Q) $2\sin^{-1}(x) = ?$; $x \in [-1, 1]$~~

~~A) C1: $x = 0 \Rightarrow 2\sin^{-1}(x) = 0$~~

~~C2: $x > 0$. Let $\sin(\theta) = x$ where $\theta \in (0, \pi/2]$~~

~~$\Rightarrow \theta = \sin^{-1}(x)$~~

~~$\Rightarrow 2\theta = 2\sin^{-1}(x)$~~

~~Now, $\cos(\theta) = \sqrt{1-x^2} \Rightarrow \sin(2\theta) = 2x\sqrt{1-x^2}$~~

~~Since $2\theta \in (0, \pi] \Rightarrow$~~

4)

$$2\sin^{-1}(x) = \begin{cases} -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right) ; & x \in [-1, -1/\sqrt{2}] \\ \sin^{-1}\left(2x\sqrt{1-x^2}\right) ; & x \in [-1/\sqrt{2}, 1/\sqrt{2}] \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right) ; & x \in [1/\sqrt{2}, 1] \end{cases}$$

Proof:

A) C1: $x \in [-1/\sqrt{2}, 1/\sqrt{2}]$ Let $\theta = \sin^{-1}(x)$

where $\theta \in [-\pi/4, \pi/4]$

Now, $2\theta \in [-\pi/2, \pi/2]$ & $2\theta = 2\sin^{-1}(x)$

Now, $\sin(2\theta) = 2x\sqrt{1-x^2} \Rightarrow 2\theta = \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow 2\sin^{-1}(x) = \sin^{-1}(2x\sqrt{1-x^2})$

C2 : $x \in (1/\sqrt{2}, 1]$ Let $\theta = \sin^{-1}(x)$ where $\theta \in (\pi/4, \pi/2]$

Now, $2\theta \in (\pi/2, \pi] \Rightarrow (\pi - 2\theta) \in [0, \pi/2)$

Now, $\sin(2\theta) = 2x\sqrt{1-x^2} = \sin(\pi - 2\theta)$

$\Rightarrow 2\theta = \pi - \sin^{-1}(2x\sqrt{1-x^2})$

$\Rightarrow 2\sin^{-1}(x) = \pi - \sin^{-1}(2x\sqrt{1-x^2})$

C3: $x \in [-1, -1/\sqrt{2})$ Let $\theta = \sin^{-1}(x)$ where $\theta \in [-\pi/2, -\pi/4)$

Now, $2\theta \in [-\pi, -\pi/2) \Rightarrow (-\pi) - 2\theta \in (-\pi/2, 0]$

Now, $\sin(2\theta) = 2x\sqrt{1-x^2} = \sin(-\pi - 2\theta)$

$\Rightarrow 2\theta = (-\pi) - \sin^{-1}(2x\sqrt{1-x^2})$

5) $2\tan^{-1}(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$; $x \geq 0$.

Proof : Let $x = \tan(\theta)$ where $\theta \in [0, \pi/2)$

Now, $2\tan^{-1}(\tan(\theta)) = 2\tan^{-1}(x) = \cancel{2\theta}\ 2\theta$

Now, $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2(\theta)}{1+\tan^2(\theta)}\right) = \cos^{-1}(\cos(2\theta))$

where $2\theta \in [0, \pi)$

$\Rightarrow \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = 2\theta$

Hence, $2\tan^{-1}(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$; $x \geq 0$

6) $2\tan^{-1}(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$; $|x| < 1$

$2\tan^{-1}(x) = \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$; $x > 1$

Proof : C1 : $|x| < 1$; Let $\tan(\theta) = x$ where $\theta \in (-\pi/4, \pi/4)$

Now, $2\tan^{-1}(\tan(\theta)) = 2\theta = 2\tan^{-1}(x)$

Now, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan(\theta)}{1+\tan^2(\theta)}\right) = \sin^{-1}(\sin(2\theta)) = 2\theta$

as $2\theta \in (-\pi/2, \pi/2)$

Now, $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan(\theta)}{1-\tan^2(\theta)}\right) = \tan^{-1}(\tan(2\theta)) = 2\theta$

as $2\theta \in (-\pi/2, \pi/2)$

C2: $x>1$, Let $\tan(\theta)=x$ where $\theta\in(\pi/4,\pi/2)$

Now, $2\tan^{-1}(x) = 2\tan^{-1}(\tan(\theta)) = 2\theta$

Now, $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan(\theta)}{1+\tan^2(\theta)}\right) = \sin^{-1}(\sin(2\theta))$ where $2\theta\in(\pi/2,\pi)$

$\Rightarrow (\pi-2\theta)\in(0,\pi/2)$

$\Rightarrow \sin^{-1}\left(\frac{2x}{1+x^2}\right) = (\pi-2\theta) \Rightarrow 2\theta = \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Now, $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \tan^{-1}\left(\frac{2\tan(\theta)}{1-\tan^2(\theta)}\right) = \tan^{-1}(\tan(2\theta))$ where $2\theta\in(\pi/2,\pi)$

$\Rightarrow (-\pi+2\theta)\in$ ~~$(0,\pi/2)$~~ $(-\pi/2,0)$

$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2}\right) = (2\theta-\pi)$

$\Rightarrow 2\theta = \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Q) Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + 2\tan^{-1}(x)$; $x>1$.

Find $f(2013)$

A) Let $\tan(\theta)=x$ where ~~$\theta\in(\pi/4$~~ $\theta\in(\pi/4,\pi/2)$

$\Rightarrow f(x) = \sin^{-1}(\sin(2\theta)) + 2\tan^{-1}(\tan(\theta))$

$= (\pi-2\theta) + 2\theta = \pi$

7) $$\sin^{-1}(3x-4x^3) = \begin{cases} 3\sin^{-1}(x) & ; \ x \in [-1/2, 1/2] \\ \pi - 3\sin^{-1}(x) & ; \ x \in (1/2, 1] \\ -\pi - 3\sin^{-1}(x) & ; \ x \in [-1, -1/2) \end{cases}$$

Proof: Let $\sin(\theta) = x \Rightarrow \sin^{-1}(3x-4x^3) = \sin^{-1}(\sin(3\theta))$
where $\theta \in [-\pi/2, \pi/2]$

$$\Rightarrow \sin^{-1}(3x-4x^3) = \begin{cases} -\pi+3\theta & ; \ 3\theta \in [-3\pi/2, -\pi/2) \\ 3\theta & ; \ 3\theta \in [-\pi/2, \pi/2] \\ \pi - 3\theta & ; \ 3\theta \in (\pi/2, 3\pi/2] \end{cases}$$

$$\Rightarrow \sin^{-1}(3x-4x^3) = \begin{cases} -\pi - 3\sin^{-1}(x) & ; \ x \in [-1, -1/2) \\ 3\sin^{-1}(x) & ; \ x \in [-1/2, 1/2] \\ \pi - 3\sin^{-1}(x) & ; \ x \in (1/2, 1] \end{cases}$$

8) $$\cos^{-1}(4x^3-3x) = \begin{cases} 3\cos^{-1}(x) & ; \ x \in [1/2, 1] \\ 2\pi - 3\cos^{-1}(x) & ; \ x \in [-1/2, 1/2) \\ -2\pi + 3\cos^{-1}(x) & ; \ x \in [-1, -1/2) \end{cases}$$

Proof: Let $\theta = \cos^{-1}(x)$ where $\theta \in [0, \pi] \Rightarrow \cos^{-1}(4x^3-3x) = \cos^{-1}(\cos(3\theta))$

$$\Rightarrow \cos^{-1}(4x^3-3x) = \begin{cases} 3\theta & ; \ 3\theta \in [0, \pi) \\ 2\pi - 3\theta & ; \ 3\theta \in [\pi, 2\pi] \\ -2\pi + 3\theta & ; \ 3\theta \in [2\pi, 3\pi] \end{cases}$$

$$\Rightarrow \cos^{-1}(4x^3-3x) = \begin{cases} 3\cos^{-1}(x) & ; \ x \in [1/2, 1] \\ 2\pi - 3\cos^{-1}(x) & ; \ x \in [-1/2, 1/2] \\ -2\pi + 3\cos^{-1}(x) & ; \ x \in [-1, -1/2) \end{cases}$$

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Q) $\sum_{r=0}^{\infty}\left(\tan^{-1}\left(\frac{1}{1+r+r^2}\right)\right)$

Q) $\sum_{r=1}^{\infty}\left(\tan^{-1}\left(\frac{2^{(r-1)}}{1+2^{(2r-1)}}\right)\right)$

Q) $\sum_{n=1}^{\infty}\left[\tan^{-1}\left(\frac{8n}{n^4-2n^2+5}\right)\right]$

Q) $\tan\left(\sum_{r=1}^{\infty}\left(\tan^{-1}\left(\frac{4}{4r^2+3}\right)\right)\right)$

Q) $\sum_{r=1}^{n}\left(\sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r^2+r}}\right)\right)$

A) $\sum_{r=0}^{\infty}\left(\tan^{-1}\left(\frac{(r+1)-r}{1+r(r+1)}\right)\right) = \sum_{r=0}^{\infty}\left(\tan^{-1}(r+1)-\tan^{-1}(r)\right)$

$= \left[\tan^{-1}(\infty)-\tan^{-1}(0)\right] = \pi/2$

A) $\sum_{r=0}^{\infty}\left(\tan^{-1}\left(\frac{2^r-2^{(r-1)}}{1+2^r\cdot 2^{(r-1)}}\right)\right) = \sum_{r=1}^{\infty}\left(\tan^{-1}(2^r)-\tan^{-1}(2^{(r-1)})\right)$

$= \left[\tan^{-1}(\infty)-\tan^{-1}(1)\right] = \pi/4$

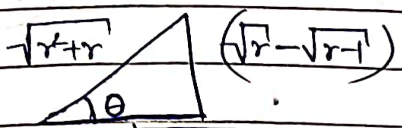
A) $\sum_{n=1}^{\infty}\left(\tan^{-1}\left(\frac{2n}{1+\left(\frac{n+1}{2}\right)^2\left(\frac{n-1}{2}\right)^2}\right)\right) = \sum_{n=1}^{\infty}\left(\tan^{-1}\left(\left(\frac{n+1}{2}\right)^2\right)-\tan^{-1}\left(\left(\frac{n-1}{2}\right)^2\right)\right)$

$= \left[\tan^{-1}(\infty)-\tan^{-1}(0)\right] = \pi/2$

A) $\tan\left(\sum_{r=1}^{\infty}\left[\tan^{-1}\left(\frac{1}{r^2+3/4}\right)\right]\right) = \tan\left(\sum_{r=1}^{\infty}\left(\tan^{-1}\left(\frac{1}{1+(r^2-1/4)}\right)\right)\right)$

$= \tan\left(\sum_{r=1}^{\infty}\left[\tan^{-1}(r+1/2)-\tan^{-1}(r-1/2)\right]\right) = \tan\left(\pi/2-\tan^{-1}(1/2)\right)$

$= 2$

A) 

$$\sqrt{r^2+r-2r+1+2\sqrt{r(r-1)}} = \sqrt{(r^2-r)+1+2\sqrt{r(r-1)}} = \sqrt{r(r-1)}+1$$

$$\Rightarrow \sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r^2+r}}\right) = \tan^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{1+\sqrt{r(r-1)}}\right)$$

$$\Rightarrow \sum_{r=1}^{n}\left(\sin^{-1}\left(\frac{\sqrt{r}-\sqrt{r-1}}{\sqrt{r(r+1)}}\right)\right) = \sum_{r=1}^{n}\left(\tan^{-1}(\sqrt{r})-\tan^{-1}(\sqrt{r-1})\right)$$

$$= \tan^{-1}(\sqrt{n})$$

★ To solve <u>ANY</u> such Q, use

$$\boxed{\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}(x)-\tan^{-1}(y)}$$